

QUASISTATIC CONTACT PROBLEM WITH UNILATERAL CONSTRAINT FOR ELASTIC-VISCOPLASTIC MATERIALS

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Abstract. This paper consists of two parts. In the first part we prove the unique solvability for the abstract variational-hemivariational inequality with history-dependent operator. The proof is based on the existing result for the static variational-hemivariational inequality and a fixed point argument. In the second part, we consider a mathematical model which describes quasistatic frictional contact between a deformable body and a rigid foundation. In the model the material behaviour is modelled by an elastic-viscoplastic constitutive law. The contact is described with a normal damped response, unilateral constraint and memory term. In the analysis of this model we use the abstract result from the first part of the paper.

Key words: variational-hemivariational inequality, history-dependent operator, frictional contact, elastic-viscoplastic material, normal damped response

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1 Introduction

Many mechanical problems involving nonmonotone, multivalued relations between stresses and strains, between reactions and displacements or between generalized forces and fluxes. These relations expressed in terms of nonconvex superpotentials (cf. [15, 16]) lead to hemivariational inequalities. Let us add, that the nonconvex superpotentials (cf. [5]) generalize the notion of convex superpotential introduced by Moreau [12]. The convex superpotentials describe monotone possibly multivalued mechanical laws and they lead to variational inequalities. The variational-hemivariational inequalities were introduced by Panagiotopoulos and they represent a special class of inequalities, in which both convex and nonconvex functions occur. These type of inequalities are a useful tool in the study of nonsmooth variational problems with constraints and boundary value problems with discontinuous nonlinearities. The results associated with variational-hemivariational inequalities and its applications can be found in the monographs, e.g. [2, 7, 8, 13, 14].

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The aim of this paper is to study the existence and uniqueness of the solution of the variational-hemivariational inequality with history-dependent operator and to apply obtained result into the analysis of a quasistatic contact problem for elastic-viscoplastic materials. It should be noted that the existence and uniqueness result for the static variational-hemivariational inequality without history-dependent operator is obtained by Migórski et al. in [9]. This paper generalizes the result from [9]. The first novelty in our work is that we consider the variational-hemivariational inequality defined on a bounded interval of time. The second novelty related to the special structure of the variational-hemivariational inequality which we consider. Namely, our inequality contains convex and nonconvex functionals and, moreover, it contains so-called history-dependent operator which at any moment $t \in (0, T)$, depend on the history of the solution up to the moment t . Furthermore, we present the example of a contact problem which leads to the variational-hemivariational inequality with history-dependent operator.

The rest of the paper is structured as follows. Section 2 contain notation and definitions. In Section 3 we consider the abstract problem and we prove it unique solvability. Finally, in Section 4 we apply the result obtained in Sections 3 in the analysis of the contact problem.

2 Preliminary

We introduce the notation and we recall some preliminary material which will be used in the next parts of this paper.

Let V and X are separable and reflexive Banach spaces with the duals V^* and X^* , respectively, and $K \subset V$. We consider also the space $\mathcal{V} = L^2(0, T; V)$, where $0 < T < +\infty$. Moreover, by $\mathcal{L}(V, X)$ we denote a space of linear and bounded operators with a Banach space V with values in a Banach space X with the norm $\|\cdot\|_{\mathcal{L}(V, X)}$. The duality pairing between X^* and X is denoted by $\langle \cdot, \cdot \rangle_{X^* \times X}$, whereas the duality pairing between \mathcal{V}^* and \mathcal{V} is given by $\langle u, v \rangle_{\mathcal{V}^* \times \mathcal{V}} = \int_0^T \langle u(t), v(t) \rangle_{V^* \times V} dt$ for $u \in \mathcal{V}^*, v \in \mathcal{V}$. If X is a Hilbert space thus the inner product is denoted by $(\cdot, \cdot)_X$.

We use the following concepts of the generalized directional derivative, the Clarke subdifferential and the subgradient of a convex function.

Definition 1. The generalized directional derivative (in the sense of Clarke) of a locally Lipschitz function $\varphi: X \rightarrow \mathbb{R}$ at the point $x \in X$ in the direction $v \in X$, denoted by $\varphi^0(x; v)$ is defined by

$$\varphi^0(x; v) = \limsup_{y \rightarrow x, \lambda \downarrow 0} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}.$$

Definition 2. Let $\varphi: X \rightarrow \mathbb{R}$ be a locally Lipschitz function. The Clarke generalized gradient (subdifferential) of φ at $x \in X$, denoted by $\partial\varphi(x)$, is the subset of a dual space X^* defined by $\partial\varphi(x) = \{\zeta \in X^* \mid \varphi^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X} \text{ for all } v \in X\}$.

Definition 3. Let $\varphi: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. The subdifferential $\partial\varphi$ is generally a multivalued mapping $\partial\varphi: X \rightarrow 2^{X^*}$ defined by $\partial\varphi(x) = \{x^* \in X^* \mid \langle x^*, v - x \rangle_{X^* \times X} \leq \varphi(v) - \varphi(x) \text{ for all } v \in X\}$ for $x \in X$. The elements of the set

$\partial\varphi(x)$ are called subgradients of φ in x .

In this paper by c we will denote a positive constant which can change from line to line. The following lemma is a consequence of the Banach contraction principle.

Lemma 4. *Let X be a Banach space with a norm $\|\cdot\|_X$ and $T > 0$. Let $\Lambda : L^2(0, T; X) \longrightarrow L^2(0, T; X)$ be an operator satisfying $\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\|_X^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_X^2 ds$ for every $\eta_1, \eta_2 \in L^2(0, T; X)$, a.e. $t \in (0, T)$. Then Λ has a unique fixed point in $L^2(0, T; X)$, i.e., there exists a unique $\eta^* \in L^2(0, T; X)$ such that $\Lambda\eta^* = \eta^*$.*

Now, we recall the concept of the history-dependent operator.

Definition 5. An operator $\mathcal{S} : \mathcal{V} \longrightarrow \mathcal{V}^*$ that satisfies the inequality

$$\|(\mathcal{S}u_1)(t) - (\mathcal{S}u_2)(t)\|_{V^*} \leq L_S \int_0^t \|u_1(s) - u_2(s)\|_V ds \quad (1)$$

for $u_1, u_2 \in \mathcal{V}$, for a.e. $t \in (0, T)$ with $L_S > 0$, is called *the history-dependent operator*.

The following property of the history-dependent operators will be used later.

Lemma 6. *Let $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{V} \longrightarrow \mathcal{V}^*$ be the operators which satisfy (1), then the operator $\mathcal{S} : \mathcal{V} \longrightarrow \mathcal{V}^*$ given by $(\bar{\mathcal{S}}u)(t) = (\mathcal{S}_1u)(t) + (\mathcal{S}_2u)(t)$ for $u \in \mathcal{V}$, satisfies (1).*

Proof. The proof is straightforward so we omit it. \square

Finally, we present the result which concerns the existence and uniqueness of the solution of the static variational-hemivariational inequality. Consider the following abstract problem.

Problem 7. Find an element $u \in V$ such that $u \in K$ and

$$\langle Au, v - u \rangle_{V^* \times V} + \varphi(u, v) - \varphi(u, u) + J^0(Mu; Mv - Mu) \geq \langle f, v - u \rangle_{V^* \times V} \text{ for all } v \in K.$$

We introduce the following hypotheses.

$$\left. \begin{array}{l} A : V \longrightarrow V^* \text{ is such that} \\ \text{(a) } A \text{ is pseudomonotone.} \\ \text{(b) } A \text{ is coercive, i.e., there exist } \alpha_A > 0, \beta, \beta_1 \in \mathbb{R} \text{ and } u_0 \in K \text{ such that} \\ \quad \langle Av, v - u_0 \rangle_{V^* \times V} \geq \alpha_A \|v\|_V^2 - \beta \|v\|_V - \beta_1 \text{ for all } v \in V. \\ \text{(c) } A \text{ is strongly monotone, i.e., there exists } m_A > 0 \text{ such that} \\ \quad \langle Av_1 - Av_2, v_1 - v_2 \rangle_{V^* \times V} \geq m_A \|v_1 - v_2\|_V^2 \text{ for all } v_1, v_2 \in V. \end{array} \right\} \quad (2)$$

$$\left. \begin{array}{l} \varphi : K \times K \longrightarrow \mathbb{R} \text{ is such that} \\ \text{(a) } \varphi(u, \cdot) : K \longrightarrow \mathbb{R} \text{ is convex and lower semicontinuous on } K, \text{ for all } u \in K. \\ \text{(b) there exists } \alpha_\varphi > 0 \text{ such that} \\ \quad \varphi(u_1, v_2) - \varphi(u_1, v_1) + \varphi(u_2, v_1) - \varphi(u_2, v_2) \leq \alpha_\varphi \|u_1 - u_2\|_V \|v_1 - v_2\|_V \\ \quad \text{for all } u_1, u_2, v_1, v_2 \in K. \end{array} \right\} \quad (3)$$

$$\left. \begin{aligned}
& J: X \longrightarrow \mathbb{R} \text{ is such that} \\
& \text{(a) } J \text{ is locally Lipschitz.} \\
& \text{(b) } \|\partial J(v)\|_{X^*} \leq c_0 + c_1 \|v\|_X \text{ for all } v \in X \text{ with } c_0, c_1 \geq 0. \\
& \text{(c) there exists } \alpha_J > 0 \text{ such that} \\
& \quad J^0(v_1; v_2 - v_1) + J^0(v_2; v_1 - v_2) \leq \alpha_J \|v_1 - v_2\|_X^2 \text{ for all } v_1, v_2 \in X.
\end{aligned} \right\} \quad (4)$$

$$M: V \longrightarrow X \text{ is a linear, continuous and compact operator.} \quad (5)$$

$$K \text{ is a nonempty, closed and convex subset of } V. \quad (6)$$

$$f \in V^*. \quad (7)$$

Remark 8. Hypothesis (4)(c) is used in the proof of the uniqueness of solution to hemivariational inequalities. This hypothesis is equivalent to the following condition

$$\langle z_1 - z_2, v_1 - v_2 \rangle_{X^* \times X} \geq -\alpha_J \|v_1 - v_2\|_X^2 \quad (8)$$

for all $z_i \in \partial J(v_i), z_i, v_i \in X, i = 1, 2$ with $\alpha_J > 0$. This condition is called *the relaxed monotonicity condition* for a locally Lipschitz function J . It can be proved that for a convex function, condition (4)(c), or equivalently (8), holds with $\alpha_J = 0$.

Theorem 9. Under hypotheses (2)–(7) and

$$m_A > \alpha_\varphi + \alpha_J \|M\|^2, \quad \alpha_A > 2\alpha_J \|M\|^2 \quad (9)$$

Problem 7 has a unique solution $u \in V$.

Proof. The proof of Theorem 9 is similar to the proof of Theorem 16 in [9]. \square

3 History-dependent variational-hemivariational inequality

In this section, we study an abstract variational-hemivariational inequality which contains a history-dependent operator. We start with the time-dependent version of Problem 7. To this end, we consider the operators $A: (0, T) \times V \longrightarrow V^*$, $M: V \longrightarrow X$, the functional $J: (0, T) \times X \longrightarrow \mathbb{R}$, the functions $\varphi: K \times K \longrightarrow \mathbb{R}$ and $f: (0, T) \longrightarrow V^*$. With these data we deal with the following variational-hemivariational inequality in which the time variable plays the role of parameter.

Problem 10. Find $u \in \mathcal{V}$ such that $u(t) \in K$ and

$$\begin{aligned}
& \langle A(t, u(t)), v - u(t) \rangle_{V^* \times V} + \varphi(u(t), v) - \varphi(u(t), u(t)) \\
& \quad + J^0(t, Mu(t); M(v - u(t))) \geq \langle f(t), v - u(t) \rangle_{V^* \times V}
\end{aligned} \quad (10)$$

for all $v \in K$ and a.e. $t \in (0, T)$.

In the study of Problem 10, we assume that the assumptions (3), (5) and (6) hold. Moreover, we need the following assumptions on the data.

$$\left. \begin{aligned}
 &A: (0, T) \times V \longrightarrow V^* \text{ is such that} \\
 &\text{(a) } A(\cdot, v) \text{ is measurable on } (0, T) \text{ for all } v \in V. \\
 &\text{(b) } A(t, \cdot) \text{ is strongly monotone, i.e., there exists } m_A > 0 \text{ such that} \\
 &\quad \langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_{V^* \times V} \geq m_A \|v_1 - v_2\|_V^2 \\
 &\quad \text{for all } v_1, v_2 \in V \text{ and a.e. } t \in (0, T). \\
 &\text{(c) } A(t, \cdot) \text{ is continuous on } V \text{ for a.e. } t \in (0, T). \\
 &\text{(d) } \|A(t, v)\|_{V^*} \leq a_0(t) + a_1 \|v\|_V \text{ for all } v \in V, \text{ a.e. } t \in (0, T) \\
 &\quad \text{with } a_0 \in L^2(0, T), a_0 \geq 0 \text{ and } a_1 > 0. \\
 &\text{(e) } A(t, \cdot) \text{ is coercive, i.e., there exists } \alpha_A > 0, \beta \in \mathbb{R}, \beta_1(t) \in L^2(0, T) \\
 &\quad \text{and } u_0 \in K \text{ such that } \langle A(t, v), v - u_0 \rangle_{V^* \times V} \geq \alpha_A \|v\|_V^2 - \beta \|v\|_V - \beta_1(t) \\
 &\quad \text{for all } v \in V \text{ a.e. } t \in (0, T).
 \end{aligned} \right\} \quad (11)$$

$$\left. \begin{aligned}
 &J: (0, T) \times X \rightarrow \mathbb{R} \text{ is such that} \\
 &\text{(a) } J(\cdot, v) \text{ is measurable on } (0, T) \text{ for all } v \in X. \\
 &\text{(b) } J(t, \cdot) \text{ is locally Lipschitz on } X \text{ for a.e. } t \in (0, T). \\
 &\text{(c) } \|\partial J(t, v)\|_{X^*} \leq c_0(t) + c_1 \|v\|_X \text{ for all } v \in X, \text{ a.e. } t \in (0, T) \text{ with} \\
 &\quad c_0 \in L^2(0, T), c_0, c_1 \geq 0. \\
 &\text{(d) } J(t, \cdot) \text{ or } -J(t, \cdot) \text{ is regular (in the sense of Clarke) on } X \text{ for} \\
 &\quad \text{a.e. } t \in (0, T). \\
 &\text{(e) there exists } m_J > 0 \text{ such that} \\
 &\quad J^0(t, v_1; v_2 - v_1) + J^0(t, v_2; v_1 - v_2) \leq m_J \|v_1 - v_2\|_X^2 \\
 &\quad \text{for all } v_1, v_2 \in X \text{ and a.e. } t \in (0, T).
 \end{aligned} \right\} \quad (12)$$

Moreover, we assume that

$$\left. \begin{aligned}
 &\text{(a) } f \in \mathcal{V}^*. \\
 &\text{(b) } m_A > \alpha_\varphi + m_J \|M\|^2, \quad \alpha_A > 2m_J \|M\|^2, \quad \text{where } \|M\| = \|M\|_{\mathcal{L}(V, X)}.
 \end{aligned} \right\} \quad (13)$$

We have the following existence and uniqueness result.

Theorem 11. *Under the assumptions (3), (5), (6) and (11)–(13), Problem 10 has a unique solution $u \in \mathcal{V}$.*

Proof. We use Theorem 9 for $t \in (0, T)$ fixed. Note that, from the hypothesis (11), it follows that the operator $A(t, \cdot)$ satisfies (2) for a.e. $t \in (0, T)$. From (11)(b),(c),(d) we observe, that A is monotone and hemicontinuous and bounded. Hence and from Theorem 3.69 in [10], we know that the operator $A(t, \cdot)$ is pseudomonotone, so the condition (2)(a) holds for a.e. $t \in (0, T)$. Moreover, for a.e. $t \in (0, T)$, the condition (11)(e) implies (2)(b). We also see, that from the hypothesis (12) and (5), it follows that the function $J(t, \cdot)$ satisfies (4) for a.e. $t \in (0, T)$. Note that, the assumption (13)(b) implies the assumption (9) with $\alpha_J = m_J$. Hence, exploiting Theorem 9, we deduce that, for a.e. $t \in (0, T)$, Problem 10 has a unique solution $u(t) \in K$.

Now, we prove that the function $t \mapsto u(t)$ is measurable on $(0, T)$. Let $g \in V^*$ be given and $u(t) \in V$ be the unique solution of the inequality (10). We claim that the solution u depends continuously on the right-hand side g , for a.e. $t \in (0, T)$. Namely, let $g_1, g_2 \in V^*$ and $u_1(t), u_2(t) \in K$ be the corresponding solutions to (10). Then

$$\begin{aligned} \langle A(t, u_1(t)), v - u_1(t) \rangle_{V^* \times V} + \varphi(u_1(t), v) - \varphi(u_1(t), u_1(t)) \\ + J^0(t, Mu_1(t); M(v - u_1(t))) \geq \langle g_1, v - u_1(t) \rangle_{V^* \times V} \end{aligned} \quad (14)$$

and

$$\begin{aligned} \langle A(t, u_2(t)), v - u_2(t) \rangle_{V^* \times V} + \varphi(u_2(t), v) - \varphi(u_2(t), u_2(t)) \\ + J^0(t, Mu_2(t); M(v - u_2(t))) \geq \langle g_2, v - u_2(t) \rangle_{V^* \times V} \end{aligned} \quad (15)$$

for all $v \in K$ and a.e. $t \in (0, T)$.

We put $v = u_2(t)$ into (14) and $v = u_1(t)$ into (15). Adding the obtained inequalities, we get

$$\begin{aligned} \langle A(t, u_1(t)) - A(t, u_2(t)), u_1(t) - u_2(t) \rangle_{V^* \times V} - (\varphi(u_1(t), u_2(t)) - \varphi(u_1(t), u_1(t)) \\ + \varphi(u_2(t), u_1(t)) - \varphi(u_2(t), u_2(t))) - (J^0(t, Mu_1(t); M(u_2(t) - u_1(t))) \\ + J^0(t, Mu_2(t); M(u_1(t) - u_2(t)))) \leq \langle g_1 - g_2, u_1(t) - u_2(t) \rangle_{V^* \times V}. \end{aligned}$$

From this, conditions (3)(b), (11)(b) and (12)(e), we have

$$\begin{aligned} m_A \|u_1(t) - u_2(t)\|_V^2 - \alpha_\varphi \|u_1(t) - u_2(t)\|_V^2 - m_J \|M\|^2 \|u_1(t) - u_2(t)\|_V^2 \\ \leq \|g_1 - g_2\|_{V^*} \|u_1(t) - u_2(t)\|_V. \end{aligned} \quad (16)$$

Exploiting (13)(b), we deduce that

$$\|u_1(t) - u_2(t)\|_V \leq c \|g_1 - g_2\|_{V^*} \quad \text{for a.e. } t \in (0, T). \quad (17)$$

Hence, we conclude that the mapping $\psi: V^* \ni g \mapsto u(t) \in V$ is continuous for a.e. $t \in (0, T)$, which proves the claim. By the condition (13)(a) we know that the function $f: [0, T] \rightarrow V^*$ is measurable. From Lemma 2.27(iii) in [10], we have that $\psi \circ f: [0, T] \rightarrow V$ is measurable. So, the solution $u(t)$ of Problem 10 is measurable on $(0, T)$.

Next, we prove that the solution of Problem 10 satisfies $u \in \mathcal{V}$. Let $v_0 \in K$. Thus, from the inequality (10), we get

$$\begin{aligned} \langle A(t, u(t)) - A(t, v_0), v_0 - u(t) \rangle_{V^* \times V} \leq \langle A(t, v_0), v_0 - u(t) \rangle_{V^* \times V} + \varphi(u(t), v_0) \\ - \varphi(u(t), u(t)) + J^0(t, Mu(t); M(v_0 - u(t))) + \langle f(t), v_0 - u(t) \rangle_{V^* \times V}. \end{aligned} \quad (18)$$

Now, we show the estimations which are needed in the next part of proof. Choosing $u_1 = u(t), u_2 = v_0, v_1 = u(t), v_2 = v_0$ in (3)(b), we obtain

$$\varphi(u(t), v_0) - \varphi(u(t), u(t)) + \varphi(v_0, u(t)) - \varphi(v_0, v_0) \leq \alpha_\varphi \|u(t) - v_0\|_V^2$$

and

$$\varphi(u(t), v_0) - \varphi(u(t), u(t)) \leq -\varphi(v_0, u(t)) + \varphi(v_0, v_0) + \alpha_\varphi \|u(t) - v_0\|_V^2. \quad (19)$$

Since $\varphi(u, \cdot)$ is convex and lower semicontinuous for $u \in K$, it admits an affine minorant (cf. Proposition 5.2.25 in [6]), i.e., there are $l_{v_0} \in V^*$ and $b_{v_0} \in \mathbb{R}$ such that $\varphi(v_0, v) \geq \langle l_{v_0}, v \rangle_{V^* \times V} + b_{v_0}$ for all $v \in V$. Using this inequality, we deduce that $-\varphi(v_0, u) \leq \|l_{v_0}\|_{V^*} \|u\|_V - b_{v_0}$ for all $u \in V$, so

$$\begin{aligned} \varphi(v_0, v_0) - \varphi(v_0, u(t)) &\leq \|l_{v_0}\|_{V^*} \|u(t)\|_V - b_{v_0} + \varphi(v_0, v_0) \leq \\ &\|l_{v_0}\|_{V^*} \|v_0 - u(t)\|_V + \|l_{v_0}\|_{V^*} \|v_0\|_V + |b_{v_0}| + |\varphi(v_0, v_0)|. \end{aligned} \quad (20)$$

Combining (19) and (20), we conclude that

$$\begin{aligned} \varphi(u(t), v_0) - \varphi(u(t), u(t)) &\leq \|l_{v_0}\|_{V^*} \|v_0 - u(t)\|_V + \|l_{v_0}\|_{V^*} \|v_0\|_V + |b_{v_0}| \\ &+ |\varphi(v_0, v_0)| + \alpha_\varphi \|u(t) - v_0\|_V^2. \end{aligned} \quad (21)$$

On the other hand, from Proposition 3.23(iii) in [10], the Cauchy-Schwartz inequality and the condition (12)(c), we obtain

$$\begin{aligned} J^0(t, Mu(t); M(v_0 - u(t))) &= \max\{\langle \zeta(t), M(v_0 - u(t)) \rangle_{X^* \times X} \mid \zeta(t) \in \partial J(t, Mu(t))\} \\ &\leq \|\partial J(t, Mu(t))\|_{X^*} \|M(v_0 - u(t))\|_X \\ &\leq (c_0(t) + c_1 \|M\| \|u(t)\|_V) \|M\| \|v_0 - u(t)\|_V. \end{aligned} \quad (22)$$

Using conditions (11)(b),(d) and estimates (21), (22) into the inequality (18), we see that

$$\begin{aligned} (m_A - \alpha_\varphi) \|v_0 - u(t)\|_V^2 &\leq (a_0(t) + a_1 \|v_0\|_V) \|v_0 - u(t)\|_V + \|l_{v_0}\|_{V^*} \|v_0 - u(t)\|_V \\ &+ \|l_{v_0}\|_{V^*} \|v_0\|_V + |b_{v_0}| + |\varphi(v_0, v_0)| + (c_0(t) + c_1 \|M\| \|u(t)\|_V) \|M\| \|v_0 - u(t)\|_V \\ &+ \|f(t)\|_{V^*} \|v_0 - u(t)\|_V. \end{aligned}$$

Hence, from the condition (13)(b) and the elementary property, namely, $x^2 \leq ax + b$ imply $x^2 \leq a^2 + b$ for $x, a, b \geq 0$, we have

$$\begin{aligned} \|v_0 - u(t)\|_V^2 &\leq c^2 (a_0(t) + a_1 \|v_0\|_V + \|l_{v_0}\|_{V^*} + \|M\| c_0(t) + c_1 \|M\|^2 \|u(t)\|_V + \|f(t)\|_{V^*})^2 \\ &+ \|l_{v_0}\|_{V^*} \|v_0\|_V + |b_{v_0}| + |\varphi(v_0, v_0)|. \end{aligned}$$

From this and inequality $\|u(t)\|^2 \leq 2 \|u(t) - v_0\|_V^2 + 2 \|v_0\|_V^2$, we conclude that

$$\begin{aligned} \|u(t)\|_V^2 &\leq 2 [c^2 (a_0(t) + a_1 \|v_0\|_V + \|l_{v_0}\|_{V^*} + \|M\| c_0(t) + c_1 \|M\|^2 \|u(t)\|_V + \|f(t)\|_{V^*})^2 \\ &+ \|l_{v_0}\|_{V^*} \|v_0\|_V + |b_{v_0}| + |\varphi(v_0, v_0)|] + 2 \|v_0\|_V^2. \end{aligned}$$

Thus, the inequality $(\sum_{i=1}^m a_i)^2 \leq m \sum_{i=1}^m a_i^2$ for $a_i \geq 0$ implies that

$$\|u(t)\|_V^2 \leq c_1^2 (a_0^2(t) + c_0^2(t) + \|f(t)\|_{V^*}^2 + c_2^2) + c_3,$$

where $c_1, c_2, c_3 \geq 0$ are constants. Integrating the last inequality over the interval $(0, T)$, we deduce that $\|u\|_{\mathcal{V}} \leq c$. Hence and the fact that $f \in \mathcal{V}^*$, we deduce that $u \in \mathcal{V}$. The proof is finished. \square

In the next problem, in contrast to Problem 10, the convex function $\tilde{\varphi}$ depends on the three arguments which follows directly from the application (cf. Section 4).

Problem 12. Find $u \in \mathcal{V}$ such that $u(t) \in K$ and

$$\begin{aligned} \langle A(t, u(t)), v - u(t) \rangle_{V^* \times V} + \tilde{\varphi}((\mathcal{S}u)(t), u(t), v) - \tilde{\varphi}((\mathcal{S}u)(t), u(t), u(t)) \\ + J^0(t, Mu(t); M(v - u(t))) \geq \langle f(t), v - u(t) \rangle_{V^* \times V} \end{aligned} \quad (23)$$

for all $v \in K$ and a.e. $t \in (0, T)$.

The inequality (23) represents a *variational-hemivariational inequality with history-dependent operator*.

As before, we assume that the operators \mathcal{S}, A and the functions J, f satisfy conditions (1), (11), (12) and (13)(a), respectively. Additionally, we assume that the function

$$\left. \begin{aligned} &\tilde{\varphi}: V^* \times K \times K \longrightarrow \mathbb{R} \text{ is such that} \\ &\text{(a) } \tilde{\varphi}(w, u, \cdot): K \longrightarrow \mathbb{R} \text{ is convex and lower semicontinuous on } K, \text{ for all} \\ &\quad w \in V^*, u \in K. \\ &\text{(b) there exists } \alpha_{\tilde{\varphi}} > 0 \text{ such that} \\ &\quad \tilde{\varphi}(w_1, u_1, v_2) - \tilde{\varphi}(w_1, u_1, v_1) + \tilde{\varphi}(w_2, u_2, v_1) - \tilde{\varphi}(w_2, u_2, v_2) \\ &\quad \leq \alpha_{\tilde{\varphi}} (\|u_1 - u_2\|_V + \|w_1 - w_2\|_V) \|v_1 - v_2\|_V \text{ for all } w_1, w_2 \in V^*, \\ &\quad u_1, u_2, v_1, v_2 \in K. \end{aligned} \right\} \quad (24)$$

Theorem 13. Under the assumptions (1), (11)–(13) and (24), Problem 12 has a unique solution $u \in \mathcal{V}$.

Proof. Let $\eta \in \mathcal{V}^*$ be fixed and we consider the following auxiliary problem.

Problem 14. Find $u_\eta \in \mathcal{V}$ such that $u_\eta(t) \in K$ and

$$\begin{aligned} \langle A(t, u_\eta(t)), v - u_\eta(t) \rangle_{V^* \times V} + \tilde{\varphi}(\eta(t), u_\eta(t), v) - \tilde{\varphi}(\eta(t), u_\eta(t), u_\eta(t)) \\ + J^0(t, Mu_\eta(t); M(v - u_\eta(t))) \geq \langle f(t), v - u_\eta(t) \rangle_{V^* \times V} \end{aligned}$$

for all $v \in K$ and a.e. $t \in (0, T)$.

Let $\phi_\eta: K \times K \longrightarrow \mathbb{R}$ be defined by $\phi_\eta(w, v) = \tilde{\varphi}(\eta(t), w, v)$ for all $w, v \in K$ and for a.e. $t \in (0, T)$. We show that function ϕ_η satisfies (3). It is easy to see that function $\phi_\eta(u, \cdot)$ satisfies (24)(a) for a.e. $t \in (0, T)$ and for all $u \in K$. Moreover, using (24)(b), we infer that

$$\begin{aligned} \phi_\eta(u_1, v_2) - \phi_\eta(u_1, v_1) + \phi_\eta(u_2, v_1) - \phi_\eta(u_2, v_2) &= \tilde{\varphi}(\eta(t), u_1, v_2) \\ &- \tilde{\varphi}(\eta(t), u_1, v_1) + \tilde{\varphi}(\eta(t), u_2, v_1) - \tilde{\varphi}(\eta(t), u_2, v_2) \leq \alpha_{\tilde{\varphi}} \|u_1 - u_2\|_V \|v_1 - v_2\|_V \end{aligned} \quad (25)$$

for all $u_1, u_2, v_1, v_2 \in K$, and a.e. $t \in (0, T)$. So, the condition (3)(b) holds with $\alpha_\phi = \alpha_{\tilde{\varphi}}$.

Hence and from Theorem 11, we deduce that Problem 14 has the unique solution $u_\eta \in \mathcal{V}$. Next, we define the operator $\Lambda: \mathcal{V}^* \longrightarrow \mathcal{V}^*$ by $\Lambda\eta = \mathcal{S}u_\eta$ for all $\eta \in \mathcal{V}^*$, where $u_\eta \in \mathcal{V}$ is the solution to Problem 14.

Claim 15. The operator Λ has a unique fixed point $\eta^* \in \mathcal{V}^*$.

Let $\eta_1, \eta_2 \in \mathcal{V}^*$, $t \in (0, T)$ and let $u_i = u_{\eta_i} \in \mathcal{V}$ for $i = 1, 2$, be the corresponding solutions to Problem 14. We put into the inequality in Problem 14, $v = u_2(t) - u_1(t)$ and $v = u_1(t) - u_2(t)$,

respectively. Thus,

$$\begin{aligned} \langle A(t, u_1(t)), u_2(t) - u_1(t) \rangle_{V^* \times V} + \tilde{\varphi}(\eta_1(t), u_1(t), u_2(t)) - \tilde{\varphi}(\eta_1(t), u_1(t), u_1(t)) \\ + J^0(t, Mu_1(t); M(u_2(t) - u_1(t))) \geq \langle f(t), u_2(t) - u_1(t) \rangle_{V^* \times V} \end{aligned}$$

and

$$\begin{aligned} \langle A(t, u_2(t)), u_1(t) - u_2(t) \rangle_{V^* \times V} + \tilde{\varphi}(\eta_2(t), u_2(t), u_1(t)) - \tilde{\varphi}(\eta_2(t), u_2(t), u_2(t)) \\ + J^0(t, Mu_2(t); M(u_1(t) - u_2(t))) \geq \langle f(t), u_1(t) - u_2(t) \rangle_{V^* \times V}. \end{aligned}$$

Adding obtained inequalities, we have

$$\begin{aligned} \langle A(t, u_1(t)) - A(t, u_2(t)), u_1(t) - u_2(t) \rangle_{V^* \times V} - \left(J^0(t, Mu_1(t); M(u_2(t) - u_1(t))) \right. \\ \left. + J^0(t, Mu_2(t); M(u_1(t) - u_2(t))) \right) \leq \tilde{\varphi}(\eta_1(t), u_1(t), u_2(t)) - \tilde{\varphi}(\eta_1(t), u_1(t), u_1(t)) \\ + \tilde{\varphi}(\eta_2(t), u_2(t), u_1(t)) - \tilde{\varphi}(\eta_2(t), u_2(t), u_2(t)). \end{aligned}$$

Using (11)(b), (12)(e) and (24)(b), we get

$$\begin{aligned} m_A \|u_1(t) - u_2(t)\|_V^2 - \left(\alpha_{\tilde{\varphi}} \|u_1(t) - u_2(t)\|_V^2 + m_J \|M\|^2 \|u_1(t) - u_2(t)\|_V^2 \right) \\ \leq \alpha_{\tilde{\varphi}} \|\eta_1(t) - \eta_2(t)\|_{V^*} \|u_1(t) - u_2(t)\|_V. \end{aligned}$$

Hence, by the condition (13)(b) with $\alpha_{\varphi} = \alpha_{\tilde{\varphi}}$, we obtain

$$\|u_1(t) - u_2(t)\|_V \leq c \|\eta_1(t) - \eta_2(t)\|_{V^*}$$

which together with the inequality (cf. (1))

$$\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\|_{V^*} = \|(\mathcal{S}u_1)(t) - (\mathcal{S}u_2)(t)\|_{V^*} \leq L_S \int_0^t \|u_1(s) - u_2(s)\|_V ds$$

imply that $\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\|_{V^*} \leq cL_S \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V^*} ds$. From the last inequality and the Hölder inequality, we conclude that

$$\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\|_{V^*}^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V^*}^2 ds \quad \text{for a.e. } t \in (0, T).$$

Applying Lemma 4, we deduce that there exists a unique $\eta^* \in \mathcal{V}^*$ such that $\Lambda\eta^* = \eta^*$, which concludes the proof of the claim.

Now, we continue the proof of Theorem 13.

Existence. Let $\eta^* \in \mathcal{V}^*$ be the fixed point of the operator Λ (cf. Claim 15). We put $\eta = \eta^*$ in Problem 14 and since $\eta^* = \Lambda\eta^* = \mathcal{S}u_{\eta^*}$, we see that $u_{\eta^*} \in \mathcal{V}$ is a solution to Problem 12.

Uniqueness. Here, we use the Gronwall-type argument. Let $u_1, u_2 \in \mathcal{V}$ be solutions to Problem 12 and $t \in (0, T)$. Then, proceeding similarly as in the proof of Theorem 11, we see that

$$\begin{aligned} \langle A(t, u_1(t)) - A(t, u_2(t)), u_1(t) - u_2(t) \rangle_{V^* \times V} - \left(J^0(t, Mu_1(t); M(u_2(t) - u_1(t))) \right. \\ \left. + J^0(t, Mu_2(t); M(u_1(t) - u_2(t))) \right) \leq \tilde{\varphi}((\mathcal{S}u_1)(t), u_1(t), u_2(t)) - \tilde{\varphi}((\mathcal{S}u_1)(t), u_1(t), u_1(t)) \\ + \tilde{\varphi}((\mathcal{S}u_2)(t), u_2(t), u_1(t)) - \tilde{\varphi}((\mathcal{S}u_2)(t), u_2(t), u_2(t)). \end{aligned}$$

Using conditions (11)(b), (12)(e) and (24)(b), we get

$$\begin{aligned} m_A \|u_1(t) - u_2(t)\|_V^2 - (\alpha_{\tilde{\varphi}} + m_J \|M\|^2) \|u_1(t) - u_2(t)\|_V^2 \\ \leq \alpha_{\tilde{\varphi}} \|(\mathcal{S}u_1)(t) - (\mathcal{S}u_2)(t)\|_{V^*} \|u_1(t) - u_2(t)\|_V. \end{aligned}$$

Next, from (1) and (13)(b), we have $\|u_1(t) - u_2(t)\|_V \leq c \int_0^t \|u_1(s) - u_2(s)\|_V ds$ for a.e. $t \in (0, T)$. Using the Gronwall inequality, we obtain $\|u_1(t) - u_2(t)\|_V = 0$ for a.e. $t \in (0, T)$, which implies that $u_1(t) = u_2(t)$ for a.e. $t \in (0, T)$. The proof of the theorem is complete. \square

4 Quasistatic elastic-viscoplastic contact problem with normal damped response, unilateral constraint and memory term

In this section we use the results obtained in Section 3 into the study the elastic-viscoplastic contact problem with normal damped response, unilateral constraint and memory term.

The physical setting is as follows. An elastic-viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$ in applications. The boundary Γ of the domain Ω is Lipschitz continuous and it is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 with $\text{meas}(\Gamma_1) > 0$. The body is subject to the action of body forces of density f_0 and surface tractions of density f_2 which act on Γ_2 . We assume that the body is clamped on Γ_1 and it is in contact on Γ_3 with a rigid foundation. Furthermore the mechanical process is quasistatic and we study it in the time interval $[0, T]$ with $T > 0$.

We use the notation \mathbb{R}^d and \mathbb{S}^d for the d -dimensional real linear space and the space of second order symmetric tensors on \mathbb{R}^d , respectively, which are equipped with the following canonical inner products and norms

$$\begin{aligned} u \cdot v = u_i v_i, \quad \|v\|_{\mathbb{R}^d} = (v \cdot v)^{\frac{1}{2}} \quad \text{for all } u = (u_i), v = (v_i) \in \mathbb{R}^d, \\ \sigma : \tau = \sigma_{ij} \tau_{ij}, \quad \|\tau\|_{\mathbb{S}^d} = (\tau : \tau)^{\frac{1}{2}} \quad \text{for all } \sigma = (\sigma_{ij}), \tau = (\tau_{ij}) \in \mathbb{S}^d, \end{aligned}$$

where the indices i and j run between 1 and d . Let us add, that the summation convention over repeated indices is used. Let $u' = \frac{\partial u}{\partial t}$ represent the velocity field and let $\text{Div} \sigma = (\sigma_{ij,j})$ be the divergence operator. We use the standard notation for the Lebesgue and Sobolev spaces and we introduce the following Hilbert spaces

$$H = L^2(\Omega; \mathbb{R}^d) = \{v = (v_i) \mid v_i \in L^2(\Omega), 1 \leq i \leq d\},$$

$$\mathcal{H} = L^2(\Omega; \mathbb{S}^d) = \{\tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega), 1 \leq i, j \leq d\}, \quad \mathcal{H}_1 = \{\tau \in \mathcal{H} \mid \text{Div} \tau \in H\}.$$

It is worth mentioning, that the Hilbert space, presented above, are equipped with the canonical inner products

$$(u, v)_H = \int_{\Omega} u \cdot v \, dx, \quad (\sigma, \tau)_{\mathcal{H}} = \int_{\Omega} \sigma : \tau \, dx, \quad (\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (\text{Div} \sigma, \text{Div} \tau)_H$$

and the associated norms

$$\|v\|_H = \left(\int_{\Omega} (\|v(x)\|_{\mathbb{R}^d})^2 dx \right)^{\frac{1}{2}}, \quad \|\tau\|_{\mathcal{H}} = \left(\int_{\Omega} \|\tau(x)\|_{\mathbb{S}^d}^2 dx \right)^{\frac{1}{2}}, \quad \|\tau\|_{\mathcal{H}_1} = \|\tau\|_{\mathcal{H}} + \|\operatorname{Div} \tau\|_H,$$

respectively.

We consider also the real Hilbert space for the displacement

$$V = \{v \in H^1(\Omega; \mathbb{R}^d) \mid v = 0 \text{ a.e. on } \Gamma_1 \text{ and } v_\nu = 0 \text{ a.e. on } \Gamma_3\}.$$

This space is endowed with the inner product and the associated norm given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}} \quad \text{and} \quad \|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}},$$

where $\varepsilon(u) = (\varepsilon_{ij}(u))$ such that $\varepsilon_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ is the deformation operator. Additionally, the inequality $\|v\|_{L^2(\Gamma_3; \mathbb{R}^d)} \leq c_0 \|v\|_V$ holds for all $v \in V$, where c_0 is a constant which depends on Ω , Γ_1 and Γ_3 .

Assume that ν denote the outward unit normal vector on Γ , $v \in H^1(\Omega; \mathbb{R}^d)$ and σ is a regular function. Thus, the normal and tangential components of the displacement field (stress field) on the boundary Γ , are defined by $v_\nu = v \cdot \nu$, $v_\tau = v - v_\nu \cdot \nu$ ($\sigma_\nu = (\sigma \nu) \cdot \nu$, $\sigma_\tau = \sigma \nu - \sigma_\nu \cdot \nu$). In order to derive variational formulations of the contact problems we will use the Green formula and the decomposition formula which are presented below.

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (\operatorname{Div} \sigma, v)_H = \int_{\Gamma} \sigma \nu \cdot v d\Gamma \quad \text{for all } v \in H^1(\Omega; \mathbb{R}^d). \quad (26)$$

$$\sigma \nu \cdot v = \sigma_\nu v_\nu + \sigma_\tau \cdot v_\tau. \quad (27)$$

For simplicity, we will write v instead of γv , where γ denotes the trace of v on the boundary Γ . For simplicity, we use the following notation $Q = \Omega \times (0, T)$ and $\Sigma_i = \Gamma_i \times (0, T)$ for $i = 1, 2, 3$.

We study the elastic-viscoplastic contact problem which classical formulation is the following.

Problem 16. Find a displacement field $u: Q \longrightarrow \mathbb{R}^d$ and a stress field $\sigma: Q \longrightarrow \mathbb{S}^d$ such that

$$\sigma(t) = \mathcal{A}(t, \varepsilon(u'(t))) + \mathcal{B}(t, \varepsilon(u(t))) + \int_0^t \mathcal{G}(s, \sigma(s) - \mathcal{A}(s, \varepsilon(u'(s))), \varepsilon(u(s))) ds \quad \text{in } Q, \quad (28)$$

$$\operatorname{Div} \sigma(t) + f_0(t) = 0 \quad \text{in } Q, \quad (29)$$

$$u(t) = 0 \quad \text{on } \Sigma_1, \quad (30)$$

$$\sigma(t) \nu = f_2(t) \quad \text{on } \Sigma_2, \quad (31)$$

$$-\sigma_\tau(t) \in \partial j_\tau(t, u'_\tau(t)) \quad \text{on } \Sigma_3, \quad (32)$$

$$\left. \begin{aligned} u'_\nu(t) &\leq g, \quad \sigma_\nu(t) + p(u'_\nu(t)) + \int_0^t b(t-s)(u'_\nu)^+(s) ds \leq 0 \\ (u'_\nu(t) - g) \left(\sigma_\nu(t) + p(u'_\nu(t)) + \int_0^t b(t-s)(u'_\nu)^+(s) ds \right) &= 0 \end{aligned} \right\} \quad \text{on } \Sigma_3, \quad (33)$$

$$u(0) = u_0 \text{ in } \Omega. \quad (34)$$

Let us note that equation (28) is the elastic-viscoplastic constitutive law in which \mathcal{A} is the viscosity operator, \mathcal{B} is the elasticity operator and \mathcal{G} is the viscoplasticity operator. The equilibrium equation is presented by (29). The displacement and the traction boundary conditions are expressed by (30) and (31), respectively. The conditions (32) and (33) are the friction law and the contact condition with normal compliance, unilateral constraint and memory term. The law (33) without memory term, is considered in [1]. Here, p and b represent given contact functions. Finally, (34) is the initial condition and u_0 denotes the initial displacement.

In the study of Problem 16, we need the following assumptions.

$$\left. \begin{aligned} &\mathcal{A} : Q \times \mathbb{S}^d \longrightarrow \mathbb{S}^d \text{ is an operator such that} \\ &\text{(a) } \mathcal{A}(\cdot, \cdot, \varepsilon) \text{ is measurable on } Q \text{ for all } \varepsilon \in \mathbb{S}^d. \\ &\text{(b) } \mathcal{A}(x, t, \cdot) \text{ is strongly monotone, i.e., there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ &\quad (\mathcal{A}(x, t, \varepsilon_1) - \mathcal{A}(x, t, \varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}^2 \\ &\quad \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d \text{ and a.e. } (x, t) \in Q. \\ &\text{(c) } \mathcal{A}(x, t, \cdot) \text{ is continuous on } \mathbb{S}^d, \text{ for a.e. } (x, t) \in Q. \\ &\text{(d) } \|\mathcal{A}(x, t, \varepsilon)\|_{\mathbb{S}^d} \leq \bar{a}_0(x, t) + \bar{a}_1 \|\varepsilon\|_{\mathbb{S}^d} \text{ for all } \varepsilon \in \mathbb{S}^d \text{ and a.e. } (x, t) \in Q \\ &\quad \text{with } \bar{a}_0 \in L^2(Q), \bar{a}_0 \geq 0 \text{ and } \bar{a}_1 > 0. \\ &\text{(e) there exists } \alpha_{\mathcal{A}} > 0 \text{ such that } \mathcal{A}(x, t, \varepsilon) : \varepsilon \geq \alpha_{\mathcal{A}} \|\varepsilon\|_{\mathbb{S}^d}^2 \\ &\quad \text{for all } \varepsilon \in \mathbb{S}^d \text{ and a.e. } (x, t) \in Q. \end{aligned} \right\} \quad (35)$$

$$\left. \begin{aligned} &\mathcal{B} : Q \times \mathbb{S}^d \longrightarrow \mathbb{S}^d \text{ is an operator such that} \\ &\text{(a) } \mathcal{B}(\cdot, \cdot, \varepsilon) \text{ is measurable on } Q \text{ for all } \varepsilon \in \mathbb{S}^d \text{ and } \mathcal{B}(\cdot, \cdot, 0) \in L^2(Q; \mathbb{S}^d). \\ &\text{(b) } \|\mathcal{B}(x, t, \varepsilon_1) - \mathcal{B}(x, t, \varepsilon_2)\|_{\mathbb{S}^d} \leq L_{\mathcal{B}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d} \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \\ &\quad \text{a.e. } (x, t) \in Q \text{ with } L_{\mathcal{B}} > 0. \end{aligned} \right\} \quad (36)$$

$$\left. \begin{aligned} &\mathcal{G} : Q \times \mathbb{S}^d \times \mathbb{S}^d \longrightarrow \mathbb{S}^d \text{ is an operator such that} \\ &\text{(a) } \mathcal{G}(\cdot, \cdot, \sigma, \varepsilon) \text{ is measurable on } Q \text{ for all } \sigma, \varepsilon \in \mathbb{S}^d \text{ and} \\ &\quad \mathcal{G}(\cdot, \cdot, 0, 0) \in L^2(Q; \mathbb{S}^d). \\ &\text{(b) } \|\mathcal{G}(x, t, \sigma_1, \varepsilon_1) - \mathcal{G}(x, t, \sigma_2, \varepsilon_2)\|_{\mathbb{S}^d} \leq L_{\mathcal{G}} (\|\sigma_1 - \sigma_2\|_{\mathbb{S}^d} + \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}) \text{ for} \\ &\quad \text{all } \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } (x, t) \in Q \text{ with } L_{\mathcal{G}} > 0. \end{aligned} \right\} \quad (37)$$

$$\left. \begin{aligned} &j_{\tau} : \Sigma_3 \times \mathbb{R}^d \longrightarrow \mathbb{R} \text{ is such that} \\ &\text{(a) } j_{\tau}(\cdot, \cdot, \xi) \text{ is measurable on } \Sigma_3 \text{ for all } \xi \in \mathbb{R}^d, \text{ and there exists} \\ &\quad e \in L^2(\Gamma_3; \mathbb{R}^d) \text{ such that } j_{\tau}(\cdot, \cdot, e(\cdot)) \in L^1(\Sigma_3). \\ &\text{(b) } j_{\tau}(x, t, \cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for a.e. } (x, t) \in \Sigma_3. \\ &\text{(c) } \|\partial j_{\tau}(x, t, \xi)\|_{\mathbb{R}^d} \leq \bar{c}_0(t) + \bar{c}_1 \|\xi\|_{\mathbb{R}^d} \text{ for all } \xi \in \mathbb{R}^d, \text{ a.e. } (x, t) \in \Sigma_3, \\ &\quad c_0 \in L^2(0, T) \text{ with } \bar{c}_0, \bar{c}_1 \geq 0. \\ &\text{(d) there exists } \alpha_j > 0 \text{ such that} \\ &\quad j_{\tau}^0(x, t, \xi_2; \xi_1 - \xi_2) + j_{\tau}^0(x, t, \xi_1; \xi_2 - \xi_1) \leq \alpha_j \|\xi_1 - \xi_2\|_{\mathbb{R}^d}^2 \text{ for all} \\ &\quad \xi_1, \xi_2 \in \mathbb{R}^d, \text{ and a.e. } (x, t) \in \Sigma_3. \\ &\text{(e) } j_{\tau}(x, t, \cdot) \text{ or } -j_{\tau}(x, t, \cdot) \text{ is regular on } \mathbb{R}^d, \text{ for a.e. } (x, t) \in \Sigma_3. \end{aligned} \right\} \quad (38)$$

$$\left. \begin{aligned}
& p: \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+ \text{ is such that} \\
& \text{(a) } p(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and } p(\cdot, 0) \in L^2(\Gamma_3). \\
& \text{(b) there exists } L_p > 0 \text{ such that } |p(x, r_1) - p(x, r_2)| \leq L_p |r_1 - r_2| \\
& \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3.
\end{aligned} \right\} \quad (39)$$

$$g \in L^\infty(\Gamma_3), \quad g > 0. \quad (40)$$

$$b \in L^1(0, T; L^\infty(\Gamma_3)). \quad (41)$$

$$f_0 \in L^2(0, T; L^2(\Omega; \mathbb{R}^d)), \quad f_2 \in L^2(0, T; L^2(\Gamma_2; \mathbb{R}^d)). \quad (42)$$

$$u_0 \in V \quad (43)$$

The concrete example of the function j_τ which satisfies condition (38) is as follows $j_\tau(\xi) = \|\xi\|_{\mathbb{R}^d}$ for all $\xi \in \mathbb{R}^d$. Here, for simplicity, we omit the dependence on variables (x, t) . The subdifferential of function j_τ has the form

$$\partial j_\tau(\xi) = \begin{cases} \overline{B}(0, 1) & \text{if } \xi = 0 \\ \frac{\xi}{\|\xi\|_{\mathbb{R}^d}} & \text{if } \xi \neq 0 \end{cases}$$

for all $\xi \in \mathbb{R}^d$, where $\overline{B}(0, 1)$ denotes the closed unit ball in \mathbb{R}^d . Note that the function j_τ is convex and regular. We see that (38) holds with $\bar{c}_0 = 1, \bar{c}_1 = 0$ and $\alpha_j = 0$ (cf. Section 7.4 in [10]).

Now, we provide the variational formulation of Problem 16. To this end, we introduce the set of admissible displacement fields defined by

$$K = \{v \in V \mid v_\nu \leq g \text{ a.e. on } \Gamma_3\}. \quad (44)$$

Assume that (u, σ) are sufficiently smooth functions which solve (28)–(34). Let $t \in (0, T)$ be fixed and $v \in K$. We use the Green formula (26) and the equation (29) to obtain

$$\int_{\Omega} \sigma(t) : (\varepsilon(v) - \varepsilon(u'(t))) \, dx = \int_{\Omega} f_0(t) \cdot (v - u'(t)) \, dx + \int_{\Gamma} \sigma(t) \nu \cdot (v - u'(t)) \, d\Gamma.$$

Using (30), (31) and the decomposition formula (27), we get

$$\begin{aligned}
\int_{\Omega} \sigma(t) : (\varepsilon(v) - \varepsilon(u'(t))) \, dx &= \int_{\Omega} f_0(t) \cdot (v - u'(t)) \, dx \\
&+ \int_{\Gamma_2} f_2(t) \cdot (v - u'(t)) \, d\Gamma + \int_{\Gamma_3} \left(\sigma_\nu(t)(v_\nu - u'_\nu(t)) + \sigma_\tau(t) \cdot (v_\tau - u'_\tau(t)) \right) d\Gamma.
\end{aligned} \quad (45)$$

From (33), we see that

$$\begin{aligned}
\sigma_\nu(t)(v_\nu - u'_\nu(t)) &= \left(\sigma_\nu(t) + p(u'_\nu(t)) + \int_0^t b(t-s)(u'_\nu)^+(s) \, ds \right) (v_\nu - g) \\
&+ \left(\sigma_\nu(t) + p(u'_\nu(t)) + \int_0^t b(t-s)(u'_\nu)^+(s) \, ds \right) (g - u'_\nu(t)) \\
&- \left(p(u'_\nu(t)) + \int_0^t b(t-s)(u'_\nu)^+(s) \, ds \right) (v_\nu - u'_\nu(t)) \quad \text{on } \Gamma_3.
\end{aligned} \quad (46)$$

From the contact condition (33) and the definition of set K (cf. (44)), we have

$$\sigma_\nu(t)(v_\nu - u'_\nu(t)) \geq -\left(p(u'_\nu(t)) + \int_0^t b(t-s)(u'_\nu)^+(s) ds\right)(v_\nu - u'_\nu(t)) \quad \text{on } \Gamma_3,$$

and

$$\begin{aligned} \int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u'_\nu(t)) d\Gamma &\geq \\ &- \int_{\Gamma_3} \left(p(u'_\nu(t)) + \int_0^t b(t-s)(u'_\nu)^+(s) ds\right)(v_\nu - u'_\nu(t)) d\Gamma. \end{aligned} \quad (47)$$

The definition of the Clarke subdifferential and the boundary condition (32) imply that

$$\int_{\Gamma_3} \sigma_\tau(t) \cdot (v_\tau - u'_\tau(t)) d\Gamma \geq \int_{\Gamma_3} j_\tau^0(t, u'_\tau(t); v_\tau - u'_\tau(t)) d\Gamma. \quad (48)$$

Using the definition of the space V , we note that

$$v \mapsto \int_{\Omega} f_0(t) \cdot v dx + \int_{\Gamma_2} f_2(t) \cdot v d\Gamma \quad \text{for a.e. } t \in (0, T)$$

is a linear, continuous functional on V . Therefore, we may apply the Riesz representation theorem to define the function $f: (0, T) \rightarrow V^*$ by

$$\langle f(t), v \rangle_{V^* \times V} = (f_0(t), v)_H + (f_2(t), v)_{L^2(\Gamma_2; \mathbb{R}^d)} \quad (49)$$

for all $v \in V$ and a.e. $t \in (0, T)$. Combining (28) and (45)–(49), we obtain the following variational formulation of Problem 16.

Problem 17. Find $u \in \mathcal{W}$ such that $u(t) \in K$, $\sigma \in L^2(0, T; \mathcal{H})$ and

$$\begin{aligned} \sigma(t) &= \mathcal{A}(t, \varepsilon(u'(t))) + \mathcal{B}(t, \varepsilon(u(t))) + \int_0^t \mathcal{G}(s, \sigma(s) - \mathcal{A}(s, \varepsilon(u'(s))), \varepsilon(u(s))) ds, \quad \text{a.e. } t \in (0, T) \\ (\sigma(t), \varepsilon(v) - \varepsilon(u'(t)))_{\mathcal{H}} &+ \int_{\Gamma_3} p(u'_\nu(t))(v_\nu - u'_\nu(t)) d\Gamma + \int_{\Gamma_3} \left(\int_0^t b(t-s)(u'_\nu)^+(s) ds \right) (v_\nu - u'_\nu(t)) d\Gamma \\ &+ \int_{\Gamma_3} j_\tau^0(t, u'_\tau(t); v_\tau - u'_\tau(t)) d\Gamma \geq \langle f(t), v - u'(t) \rangle_{V^* \times V} \end{aligned}$$

for all $v \in K$ and a.e. $t \in (0, T)$ with $u(0) = u_0$.

The existence and uniqueness result for Problem 17 is the following.

Theorem 18. Under the assumptions (35)–(43) and

$$m_{\mathcal{A}} > \max\{1, L_P\} + \alpha_j \|\gamma\|^2, \quad \alpha_{\mathcal{A}} > 2\alpha_j \|\gamma\|^2 \quad (50)$$

Problem 17 has a unique solution.

Proof. The proof of this theorem will be carried out in two steps.

Step 1. We need the following auxiliary result.

Lemma 19. *Assume that (36) and (37) hold. Then, for all $u \in \mathcal{V}$, there exists a unique function $\sigma^I(u) \in L^2(0, T; \mathcal{H})$ such that*

$$\sigma^I(u(t)) = \int_0^t \mathcal{G}(s, \mathcal{B}(t, \varepsilon(u(s))) + \sigma^I(u(s)), \varepsilon(u(s))) ds \quad (51)$$

for a.e. $t \in (0, T)$. Moreover, if $u_1, u_2 \in \mathcal{V}$, then

$$\|\sigma^I(u_1)(t) - \sigma^I(u_2)(t)\|_{\mathcal{H}} \leq L_{\sigma^I} \int_0^t \|u_1(s) - u_2(s)\|_V ds$$

for a.e. $t \in (0, T)$ with $L_{\sigma^I} > 0$.

Proof. The proof of the lemma is presented in Lemma 6.1 in [3]. \square

In order to formulate an equivalent form of Problem 17, we use Lemma 19. We consider the following intermediate problem.

Problem 20. Find $u \in \mathcal{W}$ such that $u(t) \in K$, $\sigma \in L^2(0, T; \mathcal{H})$ and

$$\sigma(t) = \mathcal{A}(t, \varepsilon(u'(t))) + \mathcal{B}(t, \varepsilon(u(t)) + \sigma^I(u(t))) \text{ a.e. } t \in (0, T)$$

$$\begin{aligned} & (\sigma(t), \varepsilon(v) - \varepsilon(u'(t)))_{\mathcal{H}} + \int_{\Gamma_3} p(u'_\nu(t))(v_\nu - u'_\nu(t)) d\Gamma + \int_{\Gamma_3} \left(\int_0^t b(t-s)(u'_\nu)^+(s) ds \right) (v_\nu - u'_\nu(t)) d\Gamma \\ & + \int_{\Gamma_3} j_\tau^0(t, u'_\tau(t); v_\tau - u'_\tau(t)) d\Gamma \geq \langle f(t), v - u'(t) \rangle_{V^* \times V} \end{aligned}$$

for all $v \in K$ and a.e. $t \in (0, T)$ with $u(0) = u_0$, where $\sigma^I(u) \in L^2(0, T; \mathcal{H})$ is the unique function defined in Lemma 19.

Step 2. Let $u' = w$. We define the operator $\mathcal{R} : \mathcal{V} \rightarrow \mathcal{V}$ such that

$$(\mathcal{R}w)(t) = \int_0^t w(s) ds + u_0 \text{ for } w \in \mathcal{V}, \text{ a.e. } t \in (0, T). \quad (52)$$

Hence, Problem 20 can be formulated as follows.

Problem 21. Find $w \in \mathcal{V}$ such that $w(t) \in K$, $\sigma \in L^2(0, T; \mathcal{H})$ and

$$\sigma(t) = \mathcal{A}(t, \varepsilon(w(t))) + \mathcal{B}(t, \varepsilon((\mathcal{R}w)(t)) + \sigma^I((\mathcal{R}w)(t))) \text{ a.e. } t \in (0, T) \quad (53)$$

$$\begin{aligned} & (\sigma(t), \varepsilon(v) - \varepsilon(w(t)))_{\mathcal{H}} + \int_{\Gamma_3} p(w_\nu(t))(v_\nu - w_\nu(t)) d\Gamma \\ & + \int_{\Gamma_3} \left(\int_0^t b(t-s)w_\nu^+(s) ds \right) (v_\nu - w_\nu(t)) d\Gamma + \int_{\Gamma_3} j_\tau^0(t, w_\tau(t); v_\tau - w_\tau(t)) d\Gamma \\ & \geq \langle f(t), v - w(t) \rangle_{V^* \times V} \text{ for all } v \in K \text{ and a.e. } t \in (0, T). \end{aligned} \quad (54)$$

Combining (53) and (54), we obtain the following problem.

Problem 22. Find $w \in \mathcal{V}$ such that $w(t) \in K$ and

$$\begin{aligned} & (\mathcal{A}(t, \varepsilon(w(t))), \varepsilon(v) - \varepsilon(w(t)))_{\mathcal{H}} + (\mathcal{B}(t, \varepsilon((\mathcal{R}w)(t))) + \sigma^I((\mathcal{R}w)(t)), \varepsilon(v) - \varepsilon(w(t)))_{\mathcal{H}} \\ & + \int_{\Gamma_3} p(w_\nu(t))(v_\nu - w_\nu(t)) d\Gamma + \int_{\Gamma_3} \left(\int_0^t b(t-s)w_\nu^+(s) ds \right) (v_\nu - w_\nu(t)) d\Gamma \\ & + \int_{\Gamma_3} j_\tau^0(t, w_\tau(t); v_\tau - w_\tau(t)) d\Gamma \geq \langle f(t), v - w(t) \rangle_{V^* \times V} \end{aligned}$$

for all $v \in K$ and a.e. $t \in (0, T)$.

Next, we introduce the operator $A: (0, T) \times V \longrightarrow V^*$ defined by

$$\langle A(t, u), v \rangle_{V^* \times V} = (\mathcal{A}(t, \varepsilon(u)), \varepsilon(v))_{\mathcal{H}} \quad (55)$$

for all $u, v \in V$ and a.e. $t \in (0, T)$. The operator A satisfies (11)(b)–(d) with $m_A = m_{\mathcal{A}} > 0$, $a_0(t) = \sqrt{2} \|\bar{a}_0(t)\|_{L^2(\Omega)}$, $a_1 = \sqrt{2} \bar{a}_1 > 0$ (see [10], p. 205 and [4], p. 3394). Now, we prove the property (11)(f). Let $u_0 \in K$ be given. Using the Cauchy-Schwartz inequality and the conditions (35)(d),(e), we get

$$\begin{aligned} \langle A(t, u), u - u_0 \rangle_{V^* \times V} &= (\mathcal{A}(t, \varepsilon(u)), \varepsilon(u) - \varepsilon(u_0))_{\mathcal{H}} = (\mathcal{A}(t, \varepsilon(u)), \varepsilon(u))_{\mathcal{H}} + (\mathcal{A}(t, \varepsilon(u)), -\varepsilon(u_0))_{\mathcal{H}} \\ &\geq \alpha_{\mathcal{A}} \|\varepsilon(u)\|_{\mathcal{H}}^2 - \|\mathcal{A}(t, \varepsilon(u))\|_{\mathcal{H}} \|\varepsilon(u_0)\|_{\mathcal{H}} = \alpha_{\mathcal{A}} \|u\|_V^2 - \bar{a}_1 \|u\|_V - \bar{a}_0(t) \|u_0\|_V. \end{aligned}$$

So, we conclude that (11)(f) holds with $\alpha_A = \alpha_{\mathcal{A}}$, $\beta = \bar{a}_1$ and $\beta_1(t) = \bar{a}_0(t) \|u_0\|_V$.

We also define the operators $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3: \mathcal{V} \longrightarrow \mathcal{V}^*$ by

$$\left. \begin{aligned} \langle (\mathcal{S}_1 w)(t), v \rangle_{V^* \times V} &= (\mathcal{B}(t, \varepsilon((\mathcal{R}w)(t))), \varepsilon(v))_{\mathcal{H}} \\ \langle (\mathcal{S}_2 w)(t), v \rangle_{V^* \times V} &= (\sigma^I((\mathcal{R}w)(t)), \varepsilon(v))_{\mathcal{H}}, \\ \langle (\mathcal{S}_3 w)(t), v \rangle_{V^* \times V} &= \int_{\Gamma_3} \left(\int_0^t b(t-s)w_\nu^+(s) ds \right) v_\nu d\Gamma \end{aligned} \right\} \quad (56)$$

for all $w \in \mathcal{V}$, $v \in V$ and a.e. $t \in (0, T)$. The hypotheses (36), (37), (41) and the definition (56) imply that the following inequalities hold (cf. [3]).

$$(\mathcal{B}(t, \varepsilon((\mathcal{R}w_1)(t))) - \mathcal{B}(t, \varepsilon((\mathcal{R}w_2)(t))), \varepsilon(v))_{\mathcal{H}} \leq L_{\mathcal{B}} \left(\int_0^t \|w_1(s) - w_2(s)\|_V ds \right) \|v\|_V,$$

$$(\sigma^I((\mathcal{R}w_1)(t)) - \sigma^I((\mathcal{R}w_2)(t)), \varepsilon(v))_{\mathcal{H}} \leq cT \left(\int_0^t \|w_1(s) - w_2(s)\|_V ds \right) \|v\|_V,$$

$$\begin{aligned} & \int_{\Gamma_3} \left(\int_0^t b(t-s)(w_{1\nu}^+(s) - w_{2\nu}^+(s)) ds \right) v_\nu d\Gamma \\ & \leq \|b\|_{L^1(0,T;L^\infty(\Gamma_3))} \|\gamma\|^2 \left(\int_0^t \|w_1(s) - w_2(s)\|_V ds \right) \|v\|_V \end{aligned}$$

for $w_1, w_2 \in \mathcal{V}$, $v \in V$, a.e. $t \in (0, T)$. Hence, the operators $\mathcal{S}_1, \mathcal{S}_2$ and \mathcal{S}_3 , defined by (56) satisfy (1) with $L_{\mathcal{S}_1} = L_{\mathcal{B}}, L_{\mathcal{S}_2} = cT$ and $L_{\mathcal{S}_3} = \|\gamma\|^2 \|b\|_{L^1(0,T;L^\infty(\Gamma_3))}$, respectively. Moreover, from Lemma 6, we conclude that the operator $\mathcal{S}: \mathcal{V} \rightarrow \mathcal{V}^*$ defined by $\langle (\mathcal{S}w)(t), v \rangle_{V^* \times V} = \sum_{i=1}^3 \langle (\mathcal{S}_i w)(t), v \rangle_{V^* \times V}$ for all $w \in \mathcal{V}$, $v \in V$ and a.e. $t \in (0, T)$ satisfies (1) with $L_{\mathcal{S}} = L_{\mathcal{B}} + cT + \|\gamma\|^2 \|b\|_{L^1(0,T;L^\infty(\Gamma_3))}$. Next, we define the operator $P: V \rightarrow V^*$ by

$$\langle P(u), v \rangle_{V^* \times V} = \int_{\Gamma_3} p(u_\nu) v_\nu d\Gamma \quad (57)$$

for all $u, v \in V$. From (39)(b) and the Hölder inequality, we see that

$$\begin{aligned} \langle P(u) - P(v), u - v \rangle_{V^* \times V} &\leq \int_{\Gamma_3} (p(u_\nu) - p(v_\nu))(v_\nu - u_\nu) d\Gamma \\ &\leq \|p(u_\nu) - p(v_\nu)\|_{L^2(\Gamma_3)} \|u_\nu - v_\nu\|_{L^2(\Gamma_3)} \leq L_p \|u_\nu - v_\nu\|_{L^2(\Gamma_3)} \|u_\nu - v_\nu\|_{L^2(\Gamma_3)} \\ &\leq L_p \|\gamma\|^2 \|u - v\|_V \|u - v\|_V. \end{aligned}$$

Hence, we conclude that the operator P is Lipschitz continuous with $L_P = L_p \|\gamma\|^2$.

Finally, we define the functional $J: (0, T) \times L^2(\Gamma_3; \mathbb{R}^d) \rightarrow \mathbb{R}$ by

$$J(t, u) = \int_{\Gamma_3} j_\tau(x, t, u_\tau(x)) d\Gamma \quad (58)$$

for all $u \in L^2(\Gamma_3; \mathbb{R}^d)$ and a.e. $t \in (0, T)$. Under the assumption (38) the functional $J: (0, T) \times L^2(\Gamma_3; \mathbb{R}^d) \rightarrow \mathbb{R}$ defined above satisfies (12) with $c_0 = \sqrt{2 \text{meas}(\Gamma_3)} \bar{c}_0$, $c_1 = \sqrt{2} \bar{c}_1$, $d_0 = \bar{d}_0 \geq 0$ and $m_J = \alpha_j \|\gamma\|^2$. (see [11], p. 280). Under the above notation Problem 17 can be written in the following equivalent form.

$$\left. \begin{aligned} &\text{Find } w \in \mathcal{V} \text{ such that } w(t) \in K \text{ and} \\ &\langle A(t, w(t)), v - w(t) \rangle_{V^* \times V} + \langle P(w(t)), v - w(t) \rangle_{V^* \times V} \\ &+ \langle (\mathcal{S}w)(t), v - w(t) \rangle_{V^* \times V} + J^0(t, \gamma w(t); \gamma v - \gamma w(t)) \geq \langle f(t), v - w(t) \rangle_{V^* \times V} \\ &\text{for all } v \in K \text{ and a.e. } t \in (0, T). \end{aligned} \right\} \quad (59)$$

We introduce the function $\tilde{\varphi}: V^* \times K \times K \rightarrow \mathbb{R}$ defined by

$$\tilde{\varphi}(z, u, v) = \langle z, v \rangle_{V^* \times V} + \langle u, v \rangle_{V^* \times V} \quad (60)$$

for all $z \in V^*, u, v \in K$. Hence and from the Cauchy-Schwartz inequality, we have

$$\begin{aligned} &\tilde{\varphi}(z_1, u_1, v_2) - \tilde{\varphi}(z_1, u_1, v_1) + \tilde{\varphi}(z_2, u_2, v_1) - \tilde{\varphi}(z_2, u_2, v_2) \\ &= \langle z_1 - z_2, v_2 - v_1 \rangle_{V^* \times V} + \langle u_1 - u_2, v_2 - v_1 \rangle_{V^* \times V} \\ &\leq (\|z_1 - z_2\|_{V^*} + \|u_1 - u_2\|_V) \|v_2 - v_1\|_V \end{aligned}$$

for all $z_1, z_2 \in V^*, u_1, u_2, v_1, v_2 \in K$. Thus, the condition (24) holds with $\alpha_{\tilde{\varphi}} = 1$. Using the definition of the function (60) and the fact that $M = \gamma$, Problem 59 has the following form.

$$\left. \begin{aligned} &\text{Find } w \in \mathcal{V} \text{ such that } w(t) \in K \text{ and} \\ &\langle A(t, w(t)), v - w(t) \rangle_{V^* \times V} + \tilde{\varphi}((\mathcal{S}w)(t), w(t), v) \\ &- \tilde{\varphi}((\mathcal{S}w)(t), w(t), w(t)) + J^0(t, Mw(t); Mv - Mw(t)) \geq \langle f(t), v - w(t) \rangle_{V^* \times V} \\ &\text{for all } v \in K \text{ and a.e. } t \in (0, T). \end{aligned} \right\} \quad (61)$$

We observe that the condition (50) implies (13)(b) with $m_A = m_{\mathcal{A}}$, $\alpha_A = \alpha_{\mathcal{A}}$, $\alpha_\varphi = \max\{1, L_P\}$, $m_J = \alpha_j$ and $M = \gamma$. Now, applying Theorem 13 (cf. Section 3), we deduce that there exists a unique function $w \in \mathcal{V}$ that solves (61). From this and the definitions (55), (56), (57), (58) and (60), we deduce that the pair $(w, \sigma) \in \mathcal{V} \times L^2(0, T; \mathcal{H})$ is a solution to Problem 21. Let $u(t) = (\mathcal{R}w)(t)$ for a.e. $t \in (0, T)$ and $w = u'$. Thus, we conclude that the pair $(u, \sigma) \in \mathcal{W} \times L^2(0, T; \mathcal{H})$ solves Problem 20. Hence and Lemma 19, we deduce that the pair $(u, \sigma) \in \mathcal{W} \times L^2(0, T; \mathcal{H})$ is a solution to Problem 17. The proof of the theorem is complete. \square

A couple of functions (u, σ) which satisfies (28)–(34) is called a *weak solution* to Problem 17. We conclude that, under the assumptions of Theorem 18, Problem 17 has a unique weak solution with regularity $u \in W^{1,2}(0, T; V)$ and $\sigma \in L^2(0, T; \mathcal{H})$. We observe, that the regularity of the stress field is, in fact, $\sigma \in L^2(0, T; \mathcal{H}_1)$. Indeed, using (29) and (42), we deduce that $\text{Div} \sigma \in L^2(0, T; L^2(\Omega; \mathbb{R}^d))$ and hence $\sigma \in L^2(0, T; \mathcal{H}_1)$.

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